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Comparison of Two Approaches to Integration on Separated Topological Spaces

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In this note we show that the integration theory developed by N. Bourbaki in [2] is a particular case of that developed by A. and C. Ionescu-Tulcea in [3].

Our results provide additional insight into the theory of integration on separated topological spaces.

1. Throughout this note T is a separated topological space, $\mathscr{K}$ is the set of all compact parts of T and $\mathscr{A}$ is the smallest clan containing $\mathscr{K}$.

For every upper integral N (see [3, p. 1]) and $1 \leqslant p < \infty$, we denote by $\mathscr{F}^p(T, N)$ the space of all $f\colon T \to \mathbf{R}$ for which $N(|f|^p) < \infty$ and by $\mathscr{L}^p(T, N, \mathscr{R})$ the space of all $f\colon T \to \mathbf{R}$ p-integrable with respect to $(N, \mathscr{R})$, where $\mathscr{R} \subset \mathscr{F}^1(T, N)$ has the properties (L_1), (L_2) and (L_3)(see [3, p. 3]). For every upper integral N we denote by $\bar{N}$ the essential upper integral associated to N (see [3, p. 14]).

For every measure $\mu \geqslant 0$ (see [2, p. 11]) we denote by $\mu^\bullet$ the essential upper integral and by μ^* the upper integral, corresponding to μ. For $1 \leqslant p < \infty$ we denote by $\bar{\mathscr{L}}^p(T, \mu)$ the space of all p-essentially integrable functions[1] $f\colon T \to \mathbf{R}$ (that is, μ-measurable and such that $\mu^\bullet(|f|^p) < \infty$), and by $\mathscr{L}^p(T, \mu)$ the space of all p-integrable functions $f\colon T \to \mathbf{R}$ (that is, p-essentially integrable and μ-moderated) with respect to μ. We shall see below that $\mu^\bullet$ is indeed the essential upper integral[2] associated to μ^* (in the sense of [3]).

Throughout this note we denote by μ a positive measure on T.

[1] To simplify the presentation we assume that the functions in $\mathscr{L}^p(T, \mu)$ and $\bar{\mathscr{L}}^p(T, \mu)$ take values in $\mathbf{R}$; this does not restrict the generality of the results.

[2] It is obvious that the concepts of "encombrement," as defined in [2], and of upper integral, as defined in [3], are equivalent, thus $\mu^\bullet$ and μ^* are upper integrals in the sense of [3].

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2. Let $\mathscr{R}$ be the set of all functions $f\colon T \to \mathbf{R}$ of the form $\sum_{j \in J} \lambda_j \phi_{A(j)}$, where J is finite, $\lambda_j \in \mathbf{R}$ and $A(j) \in \mathscr{K}$ for $j \in J$. Then $\mathscr{A}$ is the set of all $A \subset T$ such that $\phi_A \in \mathscr{R}$ (see [1, p. 159]). If $f \in \mathscr{R}$, then $\operatorname{Supp}(f)$ is relatively compact; it follows that f is moderated, whence $\mu^{\bullet}(|f|) = \mu^*(|f|)$. It is easy to prove that

$$\mathscr{R} \subset \mathscr{F}^p(T, \mu^{\bullet}) \cap \mathscr{F}^p(T, \mu^*)$$

for every $1 \leqslant p < \infty$ and that $\mathscr{R}$ has the properties (L_1), (L_2) and (L_3)(to establish (L_3) use Proposition 4, [2, p. 15]). Note also that every $f \in \mathscr{R}$ is μ-measurable.

Now let $K \in \mathscr{K}$, let $g\colon K \to \mathbf{R}$ be a continuous mapping and let $g^0\colon T \to \mathbf{R}$ be the mapping defined by $g^0(x) = g(x)$ for $x \in K$ and $g^0(x) = 0$ for $x \notin K$. Then for every $\varepsilon > 0$ there is $g_\varepsilon \in \mathscr{R}$ such that $\operatorname{Supp}(g_\varepsilon) \subset K$ and

$$\sup_{x \in T} |g_\varepsilon(x) - g(x)| \leqslant \varepsilon.$$

THEOREM 1. *For every* $1 \leqslant p < \infty$ *we have*:

(1.1) $\mathscr{L}^p(T, \mu^*, \mathscr{R}) = \mathscr{L}^p(T, \mu)$,

(1.2) $\mathscr{L}^p(T, \mu^{\bullet} \mathscr{R}) = \bar{\mathscr{L}}^p(T, \mu)$.

Proof. Let $f \in \mathscr{L}^p(T, \mu^*, \mathscr{R})$.Then there is a sequence (f_n) of elements of $\mathscr{R}$ which converges μ^*-almost everywhere to f (see [3, p. 4]); since every f_n is μ-measurable, so is f. Since $\mu^*(|f|^p) < \infty$, we deduce that f is moderated. Hence $f \in \mathscr{L}^p(T, \mu)$.

Conversely, let $f \in \mathscr{L}^p(T, \mu)$. For every $\varepsilon < 0$ there is $g\colon T \to \mathbf{R}$ such that $K = \operatorname{Supp}(g)$ is compact, $g|K$ is continuous and

$$\mu^*(|f - g|^p)^{1/p} \leqslant \varepsilon.$$

By the remark preceding the theorem there is $h \in \mathscr{R}$ such that

$$|g(x) - h(x)| \leqslant \left(\frac{\varepsilon}{\mu^*(K)^{1/p}}\right) \phi_K(x)$$

for $x \in T$. Then

$$\mu^*(|f - h|^p)^{1/p} \leqslant \mu^*(|f - g|^p)^{1/p} + \mu^*(|g - h|^p)^{1/p} \leqslant 2\varepsilon;$$

since $\varepsilon > 0$ was arbitrary it follows that $f \in \mathscr{L}^p(T, \mu^*, \mathscr{R})$. Hence (1.1) is proved.

The relation (1.2) is proved in exactly the same manner.

One of the main properties of the upper integrals is that of *regularity* (see [3, p. 7). We shall show now that:

THEOREM 2. *The upper integrals μ^* and $\mu^\bullet$ are regular and $\mu^\bullet = \overline{\mu^*}$.*

Proof. It is enough to show that μ^* is regular and $\mu^\bullet = \overline{\mu^*}$ (see (4), [3, p. 14]).

Let $\mathscr{I}_+$ be the set of all lower semicontinuous mappings of T into $\bar{\mathbf{R}}_+$. Let $f\colon T \to \bar{\mathbf{R}}_+$ and let $\mathscr{D}_f$ be the set of all $g\colon T \to \bar{\mathbf{R}}_+$ which are $(\mu^*, \mathscr{R})$-integrable (see [3, p. 4]) and such that $g \geqslant f$. If $\mathscr{D}_f = \varnothing$ then $\mu^*(f) = \infty$ (note that if $g \in \mathscr{I}_+$ and $\mu^*(g) < \infty$ then g is $(\mu^*, \mathscr{R})$-integrable). If $\mathscr{D}_f \neq \varnothing$ then

$$\mu^*(f) \leqslant \inf\{\mu^*(g) \mid g \in \mathscr{D}_f\} \leqslant \inf\{\mu^*(g) \mid g \in \mathscr{I}_+, g \geqslant f\} = \mu^*(f)$$

and hence

$$\mu^*(f) = \inf\{\mu^*(g) | g \in \mathscr{D}_f\}.$$

Hence μ^* is regular.

Let $f\colon T \to \mathbf{R}_+$. If $g \in \mathscr{R}_+$ then gf is moderated and

$$\mu^\bullet(f) \geqslant \sup\{\mu^\bullet(gf) | g \in \mathscr{R}_+,\ g \leqslant 1\} = \sup\{\mu^*(gf) | g \in \mathscr{R}_+,\ g \leqslant 1\};$$

but we also have:

$$\mu^\bullet(f) = \sup_{K \in \mathscr{K}} \mu^\bullet(\phi_K f) = \sup_{K \in \mathscr{K}} \mu^*(\phi_K f).$$

Since $\phi_K \in \mathscr{R}_+$ for $K \in \mathscr{K}$, we conclude:

$$\mu^\bullet(f) = \sup\{\mu^*(gf) | g \in \mathscr{R}_+,\ g \leqslant 1\} = \overline{\mu^*}(f);$$

whence $\mu^\bullet = \overline{\mu^*}$.

3. Let E be a metrizable topological vector space and let $\mathscr{R}_E$ be the set of all funtions $f\colon T \to E$ of the form $\sum_{j \in J} x_j \phi_{A(j)}$, where J is finite, $x_j \in E$ and $A(j) \in \mathscr{A}$ for $j \in J$. Every $f \in \mathscr{R}_E$ is μ-measurable.

A function $f\colon T \to E$ is $(\mu^*, \mathscr{R})$-*measurable* if given any $(\mu^*, \mathscr{R})$-integrable set $B \subset T$ there is a sequence (f_n) of elements of $\mathscr{R}_E$ which converges to f on $B - B_0$, where $B_0 \subset B$ and $\mu^*(B_0) = 0$. Replacing μ^* by $\mu^\bullet$ we obtain the definition of $(\mu^\bullet, \mathscr{R})$-measurability.[3]

Obviously, every function in $\mathscr{R}_E$ is both $(\mu^*, \mathscr{R})$-measurable and $(\mu^\bullet, \mathscr{R})$-measurable. The Egorov theorem (given in [3, p. 71] for the case when E is a Banach space) is valid also in this setting; in particular, if (f_n) is a sequence of measurable functions (in the sense defined above) which converges to f almost everywhere on every integrable set, then f is measurable.

[3] One can define measurability also for functions with values in a metrizable space (see [4, p. 204]) and generalize Theorem 3 below.

Now let $K \in \mathscr{K}$, let $g: K \to E$ be a continuous mapping and let $g^0: T \to E$ be the mapping defined by $g^0(x) = g(x)$ if $x \in K$ and $g^0(x) = 0$ if $x \notin K$. It is easy to see that there is a sequence (g_n) of elements of $\mathscr{R}_E$ which converges uniformly on T to g^0. It follows that g is both $(\mu^*, \mathscr{R})$-measurable and $(\mu^\bullet, \mathscr{R})$-measurable.

THEOREM 3. *Let $f: T \to E$. Then the following are equivalent:*

(3.1) *f is $(\mu^\bullet, \mathscr{R})$-measurable,*

(3.2) *f is $(\mu^*, \mathscr{R})$-measurable,*

(3.3) *f is μ-measurable.*

Proof. Assume that (3.1) holds. Let $B \subset T$ be a $(\mu^*, \mathscr{R})$-integrable set. Then B is also $(\mu^\bullet; \mathscr{R})$-integrable. Hence there is a sequence (f_n) of elements of $\mathscr{R}_E$ which converges to f on $B - B_0$, where $B_0 \subset B$ and $\mu^\bullet(B_0) = 0$. Since B is moderated, B_0 is moderated, whence $\mu^*(B_0) = 0$. Since B is arbitrary, (3.2) holds. Hence (3.1) $\Rightarrow$ (3.2).

Since every $K \in \mathscr{K}$ is $(\mu^*, \mathscr{R})$-integrable and since every function in $\mathscr{R}_E$ is μ-measurable we deduce that (3.2) $\Rightarrow$ (3.3).

Finally, assume that (3.3) holds. Let $A \subset T$ be a $(\mu^\bullet, \mathscr{R})$-integrable set. Then A is μ-measurable (use, for example, Theorem 1) and $\mu^\bullet(A) < \infty$. Hence there exists a set $L \subset A$ and a family $(L_j)_{j \in \mathrm{N}}$ of pairwise disjoint compact sets such that $\mu^\bullet(L) = 0$, $f|L_j$ is continuous on L_j for $j \in N$ and

$$A = L \cup \left(\bigcup_{j \in N} L_j \right).$$

If $f_n = f | (\bigcup_{0 \leqslant j < n} L_j)$ for every $n \in N$, then f_n is continuous on $\bigcup_{0 \leqslant j < n} L_j$; hence f_n^0 is $(\mu^\bullet, \mathscr{R})$-measurable (see the remark preceding the theorem). Since (f_n^0) converges to $(f|A)^0$ on $T - L$, we deduce that $(f|A)^0$ is $(\mu^\bullet, \mathscr{R})$-measurable. Since A is arbitrary, f is $(\mu^\bullet, \mathscr{R})$-measurable and hence (3.3) $\Rightarrow$ (3.1).

From now on we assume E to be a Banach space. For $1 \leqslant p < \infty$ we denote by $\mathscr{L}_E^p(T, N, \mathscr{R})$ the space of all $f: T \to E$ p-integrable with respect to $(N, \mathscr{R})$; for $f \in \mathscr{L}_E^1(T, N, \mathscr{R})$ we define

$$\int_T f \, d\mu_{(N, \mathscr{R})}$$

as in [3](the integral is actually defined in [3] only in the case of real valued functions; the generalization to the case of functions with values in a Banach space is immediate).

For $1 \leqslant p < \infty$ we denote by $\bar{\mathscr{L}}_E^p(T, \mu)$ the space of all p-essentially integrable functions $f: T \to E$ (that is, μ-measurable and such that $\mu^\bullet(\|f\|^p) < \infty$), and by $\mathscr{L}_E^p(T, \mu)$ the space of all p-integrable functions

$f: T \to E$ (that is, p-essentially integrable and μ-moderated); for $f \in \bar{\mathscr{L}}_E^1(T, \mu)$ we define

$$\int_T f\, d\mu$$

as in [2]. We have the following results:

THEOREM 4. *For every* $1 \leqslant p < \infty$ *we have*:

(4.1) $\mathscr{L}_E^p(T, \mu^*, \mathscr{R}) = \mathscr{L}_E^p(T, \mu)$,

(4.2) $\mathscr{L}_E^p(T, \mu^{\cdot}, \mathscr{R}) = \bar{\mathscr{L}}_E^p(T, \mu)$.

Proof. This follows from Theorem 6 of [3, p. 71] and Theorem 3 above. We observe that this theorem can also be obtained by the method used to prove Theorem 1.

THEOREM 5. *For every* $f \in \bar{\mathscr{L}}_E^1(T, \mu) = \mathscr{L}_E^1(T, \mu^{\cdot}, \mathscr{R})$ *we have*:

$$\int_T f\, d\mu = \int_T f\, d\mu_{(\mu^{\cdot}, \mathscr{R})}$$

Proof. The vector space $\mathscr{R}_E$ is dense in $\mathscr{L}_E^1(T, \mu^{\cdot}, \mathscr{R}) = \bar{\mathscr{L}}_E^1(T, \mu)$; hence to prove Theorem 5 it is enough to consider the case when $f \in \mathscr{R}_E$, and hence the case when $f = x \cdot \phi_A$, where $x \in E$ and $A \in \mathscr{A}$. But in this case the above integrals are both equal to $\mu^{\cdot}(A) \cdot x$.

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